Return Models - Part I
A Discrete-Time Model Without Mean Reversion

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In this white paper we will build a discrete-time return model without mean reversion. The absence of mean reversion means that the company's projected go-forward growth rates and rates of return on investment are time-independent (i.e. are constants) and sustainable in perpetuity. We will define the discrete-time variable s to be number of months from the perspective of time zero. The time variable s is an integer value between zero and infinity. This statement in equation form is...

$$s \in \{0, 1, 2, 3, ..., \infty\} \tag{1}$$

We will define enterprise value to be the discounted value of expected future after-tax operating net cash flow before debt service. We will define the debt tax shield value to be the discounted value of expected future tax savings applicable to the tax-deductibility of interest payments on debt. We will define company value to be the sum of enterprise value and debt tax shield value. This statement in equation form is...

$$\text{Company value} = \text{Enterprise value} + \text{Debt tax shield value} \tag{2}$$

We will define the variables m and n to be time in months where m and n are integer values subject to the constraint $0 \leq m < n \leq \infty$. We will define company value at the end of month m to be the discounted value of expected net cash flow to be received over the time interval $[m+1, n]$. The discrete-time valuation equation takes the following form...

$$\text{Company value at the end of month } m = \sum_{s=m+1}^{n} \text{Net cash flow in month } s \times \text{Discount factor in month } s \tag{3}$$

We will define base value at the end of month s to be a linear function of notional value at the end of month s. We will define the variable N_s to be notional value at the end of month s and the variable B_s to be base value at the end of month s. The table below presents examples of notional and base values by industry...

Table 1: Notional And Base Values By Industry

Industry	Notional Value	Base Value - Enterprise Value	Base Value - Debt Tax Shield
Banking	Tangible assets	Tangible capital	Subordinated debt
Oil and Gas	Annualized production	Tangible operating assets	Interest-bearing debt
Retail	Annualized operating revenue	Tangible operating assets	Interest-bearing debt
Technology	Annualized operating revenue	Tangible operating assets	Interest-bearing debt

The following equation defines the relationship between notional value and base value...

$$B_s = \text{constant} \times N_s \;\; \text{...where... constant} = \frac{B_0}{N_0} \tag{4}$$

To assist us in building our model we will work through the following hypothetical problem...

Our Hypothetical Problem

ABC Company is a company in the retail industry with an annualized revenue growth rate of 4% and an after-tax return on assets of 18%. Using Table 1 above we will define notional and base values to be the following...

Table 2: Definitions

Notional value	Annualized operating revenue
Base value - Enterprise Value	Tangible operating assets
Base value - Debt Tax Shield	Interest-bearing debt

The table below presents ABC Company's go-forward model assumptions...

Table 3: Model Assumptions

Description	Balance
Annualized operating revenue at time zero ($)	1,000,000
Tangible operating assets at time zero ($)	1,250,000
Interest-bearing debt principal at time zero ($)	300,000
Annualized revenue growth rate (%)	4.00
After-tax return on assets (%)	18.00
Income tax rate (%)	20.00
Debt interest rate (%)	6.00
Weighted-average cost of capital (%)	12.00

We are tasked with building a return model to answer the following questions...

Question 1: What is company value at time zero given that cash flow is expected to be received over the time interval $[1, \infty]$?

Question 2: What is company value at the end of year 5 given that cash flow is expected to cease after year 15?

Question 3: Do you see any problems with this valuation?

To calculate discounted value we will need the discrete-time, weighted-average cost of capital. We will define the variable k to be the periodic weighted-average cost of capital. Using the data in Table 3 above the equation for the periodic cost of capital is...

$$k = (1 + \text{annual rate})^{\frac{1}{12}} - 1 = (1 + 0.12)^{\frac{1}{12}} - 1 = 0.00949 \tag{5}$$

Annualized Revenue

Per Table 2 above we defined notional value to be annualized operating revenue. We will define the variable R_s to be annualized operating revenue at the end of month s and the variable g to be the periodic, discrete-time revenue growth rate. The equation for annualized revenue at the end of month s is...

$$\text{if... } R_s = R_{s-1}\,(1+g) \text{ ...then... } R_s = R_0\,(1+g)^s \text{ ...where... } g = (1 + \text{annual rate})^{\frac{1}{12}} - 1 \tag{6}$$

Using Equation (6) above the equation for the change in annualized revenue over the time interval $[s, s+1]$ is...

$$\Delta\, R_s = R_{s+1} - R_s = R_0\,(1+g)^{s+1} - R_0\,(1+g)^s = R_0\,(1+g)^s((1+g)-1) = g\,R_0\,(1+g)^s = g\,R_s \tag{7}$$

Using the data in Table 3 above notional value at time zero (R_0) and the periodic revenue growth rate (g) are...

$$R_0 = 1{,}000{,}000 \text{ ...and... } g = (1 + 0.04)^{\frac{1}{12}} - 1 = 0.00327 \tag{8}$$

Assets

Per Table 2 above we defined notional and base values to be annualized operating revenue and tangible operating assets, respectively. We will define the variable A_s to be tangible operating assets at the end of month s and the variable ϕ to be the ratio of base value (assets) to notional value (annualized revenue). Using Equations (4), (6) and (8) above and the data in Table 3 above the equation for tangible operating assets at the end of month s is...

$$A_s = \phi\,R_s \text{ ...where... } A_0 = 1{,}250{,}000 \text{ ...and... } \phi = \frac{A_0}{R_0} = \frac{1{,}250{,}000}{1{,}000{,}000} = 1.25 \tag{9}$$

To calculate net cash flow we will need an equation for incremental investment, which we will define as the change in tangible operating assets over time. Using Appendix Equation (40) below the equation for the change in assets over the time interval $[s, s+1]$ is...

$$\Delta A_s = \phi \, \Delta R_s = \phi \, g \, R_s \tag{10}$$

We will define the variable r to be the periodic, discrete-time, after-tax rate of return on assets (ROA). Using the data in Table 3 above the equation for the periodic return on assets is...

$$r = \text{annual rate} \times \frac{1}{12} = \frac{0.18}{12} = 0.0150 \tag{11}$$

We will define periodic net income to be after-tax net income before debt service realized over the time interval $[s, s+1]$. Using Equations (9) and (10) above the equation for periodic net income is...

$$\text{Periodic net income} = r \, A_s = \phi \, r \, R_0 \, (1+g)^s \tag{12}$$

We will define periodic net investment to be the change in assets over the time interval $[s, s+1]$. Using Equations (7) and (11) above the equation for periodic net investment is...

$$\text{Periodic net investment} = \phi \, g \, R_s = \phi \, g \, R_0 \, (1+g)^s \tag{13}$$

We will define periodic net cash flow to be periodic net income minus periodic net investment. Using Equations (12) and (13) above the equation for periodic net cash flow over the time interval $[s, s+1]$ is...

$$\begin{aligned} \text{Periodic net cash flow} &= \text{Periodic net income} - \text{Periodic net investment} \\ &= \phi \, r \, R_0 \, (1+g)^s - \phi \, g \, R_0 \, (1+g)^s \\ &= \phi \, (r-g) \, R_0 \, (1+g)^s \end{aligned} \tag{14}$$

We will define the variable I_s to be after-tax net income over the time interval $[s, s+1]$. Using Equations (16), (17) and (18) above the equation for after-tax interest expense is...

$$I_{m,n} = \sum_{s=m}^{n} \phi \, r \, R_0 \, (1+g)^s \tag{15}$$

Interest-Bearing Debt

Per Table 2 above we defined notional and base values to be annualized operating revenue and interest-bearing debt, respectively. We will define the variable D_s to be debt principal balance at the end of month s and the variable ϵ to be the ratio of base value (debt) to notional value (annualized revenue). Using Equations (6) and (8) above and the data in Table 3 above the equation for interest-bearing debt at the end of month s is...

$$D_s = \epsilon \, R_s \;\; \text{...where...} \;\; D_0 = 300,000 \;\; \text{...and...} \;\; \epsilon = \frac{D_0}{R_0} = \frac{300,000}{1,000,000} = 0.30 \tag{16}$$

We will define the variable i to be the periodic debt interest rate. Using the data in Table 3 above the equation for the periodic interest rate is...

$$i = \text{annual rate} \times \frac{1}{12} = \frac{0.06}{12} = 0.0050 \tag{17}$$

We will define the variable α to be the income tax rate. Using the data in Table 3 above the equation for the income tax rate is...

$$\alpha = 0.20 \tag{18}$$

We will define periodic net cash flow to be the tax savings applicable to the interest expense tax deduction. Using Equations (16), (17) and (18) above the equation for periodic net cash flow over the time interval $[s, s+1]$ is...

$$\text{Periodic net cash flow} = i \, \alpha \, D_s = \epsilon \, i \, \alpha \, R_0 \, (1+g)^s \tag{19}$$

Using Equations (9), (16), (17) and (18) above the equation for periodic after-tax interest expense is...

$$\text{Periodic interest expense} = i \, D_s \, (1-\alpha) = i \, \epsilon \, R_s \, (1-\alpha) = i \, \epsilon \, R_0 \, (1+g)^s \, (1-\alpha) \tag{20}$$

GAAP Net Income

We will define the variable G_s to be periodic GAAP net income over the time interval $[s, s+1]$. GAAP net income is defined as enterprise net income (after-tax net income before debt service) minus after-tax debt interest expense. Using Equations (12) and (20) above the equation for periodic GAAP net income is...

$$G_s = \phi \, r \, R_0 \, (1+g)^s - i \, \epsilon \, R_0 \, (1+g)^s \, (1-\alpha) = R_0 \, (1+g)^s \, (\phi \, r - i \, \epsilon \, (1-\alpha)) \tag{21}$$

We will define the variable $G_{m,n}$ to be cumulative GAAP net income realized over the time interval $[m, n]$. Using Equation (21) above the equation for cumulative GAAP net income is...

$$G_{m,n} = \sum_{m}^{n} R_0 \, (1+g)^s \, (\phi \, r - i \, \epsilon \, (1-\alpha)) = R_0 \, (\phi \, r - i \, \epsilon \, (1-\alpha)) \sum_{m}^{n} (1+g)^s \tag{22}$$

Enterprise Value

Enterprise value is defined as the discounted value at the end of month m of expected enterprise net cash flow to be received over the time interval $[m+1, n]$. We will define the variable $V_{m,n}$ to be enterprise value at the end of month m. Using Equations (3), (5) and (14) above the equation for enterprise value is...

$$\begin{aligned} V_{m,n} &= \sum_{s=m+1}^{n} \text{Periodic net cash flow at time } s \times \text{Discount factor at time } s \\ &= \sum_{s=m+1}^{n} \phi \, (r-g) \, R_0 \, (1+g)^s \, (1+k)^{-(s-m)} \\ &= \phi \, (r-g) \, R_0 \, (1+k)^m \sum_{s=m+1}^{n} (1+g)^s \, (1+k)^{-s} \\ &= \phi \, (r-g) \, R_0 \, (1+k)^m \sum_{s=m+1}^{n} \left(\frac{1+g}{1+k} \right)^s \end{aligned} \tag{23}$$

We will define the variable θ to be the ratio of one plus the periodic notional value growth rate to one plus the periodic discount rate. Using Equations (5) and (8) above the equation for this ratio is...

$$\theta = \frac{1+g}{1+k} \tag{24}$$

Using Equation (24) above we can rewrite enterprise value Equation (23) above as...

$$\begin{aligned} V_{m,n} &= \phi \, (r-g) \, R_0 \, (1+k)^m \sum_{s=m+1}^{n} \theta^s \\ &= \phi \, (r-g) \, R_0 \, (1+k)^m \, \theta^m \sum_{s=1}^{n-m} \theta^s \\ &= \phi \, (r-g) \, (1+g)^m \, R_0 \sum_{s=1}^{n-m} \theta^s \end{aligned} \tag{25}$$

Note that the solution to the polylogarithm in Equation (25) above is... [1]

$$\sum_{s=1}^{n-m} \theta^s = \frac{\theta - \theta^{(n-m)+1}}{1-\theta} \tag{26}$$

Note that when the upper bound of the summation goes to infinity then Equation (26) above becomes...

$$\lim_{n \to \infty} \sum_{s=1}^{n-m} \theta^s = \frac{\theta}{1-\theta} \;\; \text{...because...} \;\; \lim_{n \to \infty} \theta^{(n-m)+1} = 0 \;\; \text{...when...} \;\; |\theta| < 1 \tag{27}$$

Debt Tax Shield Value

Debt tax shield value is defined as the discounted value at the end of month m of expected tax savings (interest expense on debt times the income tax rate) to be received over the time interval $[m+1,n]$. We will define the variable $W_{m,n}$ to be the debt tax shield value at the end of month m. Using Equations (3), (5) and (19) above the equation for debt tax shield value is...

$$\begin{aligned}
W_{m,n} &= \sum_{s=m+1}^{n} \text{Periodic net cash flow at time } s \times \text{Discount factor at time } s \\
&= \sum_{s=m+1}^{n} \epsilon\, i\, T\, R_0\, (1+g)^s\, (1+k)^{-(s-m)} \\
&= \epsilon\, i\, T\, R_0\, (1+k)^m \sum_{s=m+1}^{n} (1+g)^s\, (1+k)^{-s} \\
&= \epsilon\, i\, T\, R_0\, (1+k)^m \sum_{s=m+1}^{n} \left(\frac{1+g}{1+k}\right)^s
\end{aligned} \tag{28}$$

Using Equation (24) above we can rewrite debt tax shield value Equation (28) above as...

$$\begin{aligned}
W_{m,n} &= \epsilon\, i\, T\, R_0\, (1+k)^m \sum_{s=m+1}^{n} \theta^s \\
&= \epsilon\, i\, T\, R_0\, (1+k)^m\, \theta^m \sum_{s=1}^{n-m} \theta^s \\
&= \epsilon\, i\, T\, (1+g)^m\, R_0 \sum_{s=1}^{n-m} \theta^s
\end{aligned} \tag{29}$$

Note that Equations (26) and (27) above are the solutions to the polylogarithm in Equation (29) above.

The Answers To Our Hypothetical Problem

The table below presents our model parameters...

Table 4: Model Parameters

N_0	Annualized operating revenue	1,000,000	Equation (8)
ϕ	Ratio of assets to annualized revenue	1.25000	Equation (9)
ϵ	Ratio of debt to annualized revenue	0.30000	Equation (16)
g	Periodic revenue growth rate	0.00327	Equation (8)
k	Periodic discount rate	0.00949	Equation (5)
r	Periodic rate of return on assets	0.01500	Equation (11)
i	Periodic debt interest rate	0.00500	Equation (17)
T	Income tax rate	0.20000	Table 3

Using the Equation (24) above and the data in Table 4 above the equation for the model parameter θ is...

$$\theta = \frac{1+g}{1+k} = \frac{1.00327}{1.00949} = 0.99384 \tag{30}$$

Using the Equations (27) and (30) above the solution to the following summation over the time interval $[1,\infty]$ is ...

$$\sum_{s=1}^{\infty} \theta^s = \frac{0.99384}{1-0.99384} = 161.34 \tag{31}$$

Using the Equations (26) and (30) above the solution to the following summation over the time interval $[61,180]$ is ...

$$\sum_{s=61}^{180-60} \theta^s = \frac{0.99384 - 0.99384^{(180-60)+1}}{1-0.99384} = 84.49 \tag{32}$$

Question 1: What is company value at time zero given that cash flow is expected to be received over the time interval $[1, \infty]$?

Using Equations (25) and (31) and the data in Table 4 above the equation for enterprise value is...

$$V_{0,\infty} = 1.25 \times (0.01500 - 0.00327) \times (1 + 0.00327)^0 \times 1,000,000 \times 161.34 = 2,366,000 \tag{33}$$

Using Equations (29) and (31) and the data in Table 4 above the equation for debt tax shield value is...

$$W_{0,\infty} = 0.30 \times 0.005 \times 0.20 \times (1 + 0.00327)^0 \times 1,000,000 \times 161.34 = 48,000 \tag{34}$$

Using Equations (33) and (34) above the equation for company value is...

$$\text{Company value} = 2,366,000 + 48,000 = 2,414,000 \tag{35}$$

Question 2: What is company value at the end of year 5 given that cash flow is expected to cease after year 15?

The summation lower and upper bounds are...

$$m = 5 \times 12 = 60 \text{ ...and... } n = 15 \times 12 = 180 \tag{36}$$

Using Equations (25) and (31) and the data in Table 4 above the equation for enterprise value is...

$$V_{0,\infty} = 1.25 \times (0.01500 - 0.00327) \times (1 + 0.00327)^{60} \times 1,000,000 \times 84.49 = 1,507,000 \tag{37}$$

Using Equations (29) and (31) and the data in Table 4 above the equation for debt tax shield value is...

$$W_{0,\infty} = 0.30 \times 0.005 \times 0.20 \times (1 + 0.00327)^{60} \times 1,000,000 \times 84.49 = 31,000 \tag{38}$$

Using Equations (37) and (38) above the equation for company value is...

$$\text{Company value} = 1,507,000 + 31,000 = 1,538,000 \tag{39}$$

Question 3: Do you see any problems with this valuation?

The revenue growth rate (4%) is at or around than the nominal rate of GDP growth, which is not a problem, but the rate of return on assets (18%) is much greater than the cost of capital (12%), which is a problem. In competitive markets the rate of return on investment (assets in our case) cannot exceed the cost of capital in the long-run.

References

[1] Gary Schurman, *Polylogarithms of Order Zero,* May, 2019

Appendix

A. Using Equations (6), (7) and (**??**) the solution to the following equation is...

$$\begin{aligned} \Delta A_s &= A_{s+1} - A_s \\ &= \phi R_{s+1} - \phi R_s \\ &= \phi (R_{s+1} - R_s) \\ &= \phi\, g\, R_s \end{aligned} \tag{40}$$